Animation

Motion models

Computer animation techniques

- 2D
  - Sprite animation
  - Morphing
  - Graphic embedding
- 3D, virtual world
  - Object model
    - Articulated model
    - Particle system
    - Deformable object
    - Hybrid
  - Rendering
    - Motion blur
  - Motion models
    - Keyframing
    - Kinematics
    - Procedural
    - Motion capture

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Keyframing

- Derived from traditional animation
- Motion outlined by specifying key object positions
- Intermediate frames determined by interpolation ('inbetweening')

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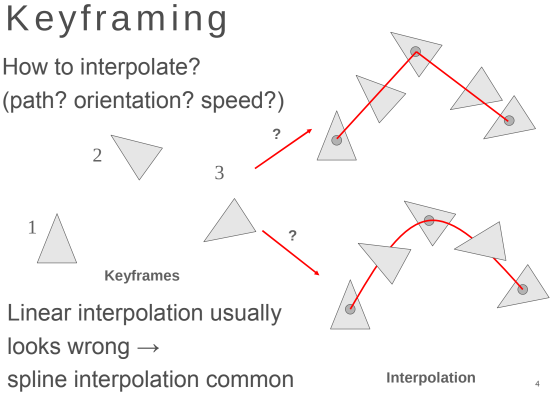
Keyframing

How to interpolate?
(path? orientation? speed?)



Linear interpolation usually looks wrong →
spline interpolation common

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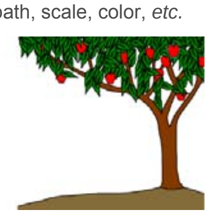
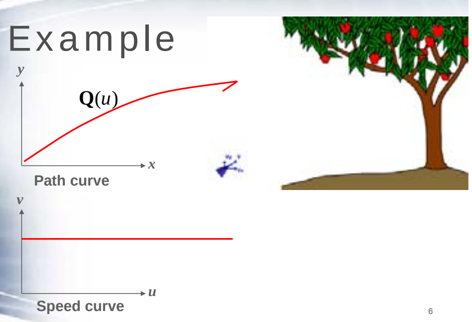
Interpolation

- Explicit scripting: specify (e.g. using spline curves) the variation of all variables to be interpolated
- Common defaults:
  - Linear interpolation for path, scale, color, *etc.*
  - Constant speed
  - Slerp or path tangent for orientation



Example



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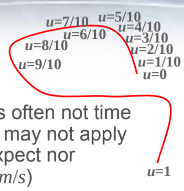
Interpolation



- Problem: curve parameter u is often not time (nor arc length) → speed $v(u)$ may not apply to the part of the curve you expect nor correspond to a ‘real’ speed (m/s)
- Reparametrize $\mathbf{Q}(u)$ to arc length s and map s to time t using $v(t)$
- Express other variables (scale, color, etc.) in s or t

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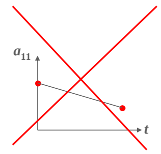
Interpolation of rotation

- A rigid body transformation can be expressed by the matrix $\mathbf{M}$

$$\mathbf{M} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & t_x \\ a_{21} & a_{22} & a_{23} & t_y \\ a_{31} & a_{32} & a_{33} & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Linear interpolation of the elements of the rotation part a will give nonsensical results!



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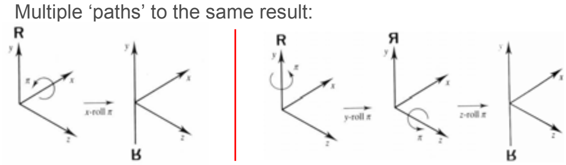
Interpolation of rotation

- The a_{ij} are dependent, only three ‘real’ variables - three angles.
- Common: Euler angle specification; three angles relative to three perpendicular axes
- Order of applying the Euler angles is important (different results)
- Multiple ‘paths’ to the same result:



- → directly interpolating Euler angles is **not** a good idea: no orientation ‘path’ control

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Understanding the rotation path

- Euler theorem: it is possible to get from one orientation to another by a single rotation θ around some axis n
- θ and n can be found using *quaternions*

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Quaternions

- A quaternion is a four-vector structure that represents orientation
- It embodies the θ and n from Euler’s theorem
- Can be used for interpolation of rotation
- Generalization of complex numbers

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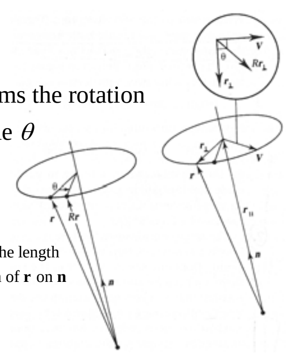
Quaternions

- operator $R = R(\theta, \mathbf{n})$ performs the rotation around unit axis $\mathbf{n}$ by an angle θ
- The vector $\mathbf{r}$ can be decomposed as $\mathbf{r}=\mathbf{r}_{\parallel} + \mathbf{r}_{\perp}$
- $\begin{cases} \mathbf{r}_{\parallel} = (\mathbf{n}\cdot\mathbf{r})\mathbf{n} \\ \mathbf{r}_{\perp} = \mathbf{r} - (\mathbf{n}\cdot\mathbf{r})\mathbf{n} \end{cases}$ ← because $\mathbf{n}\cdot\mathbf{r}$ is the length of the projection of $\mathbf{r}$ on $\mathbf{n}$
- $R\mathbf{r}_{\parallel} = \mathbf{r}_{\parallel}$



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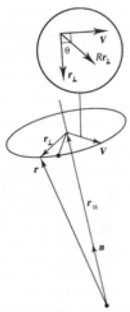
Quaternions

- $R\mathbf{r}_{\parallel} = \mathbf{r}_{\parallel}$
- $\begin{cases} \mathbf{r}_{\parallel} = (\mathbf{n}\cdot\mathbf{r})\mathbf{n} \\ \mathbf{r}_{\perp} = \mathbf{r} - (\mathbf{n}\cdot\mathbf{r})\mathbf{n} \end{cases}$



- We introduce a vector $\mathbf{v}$ perpendicular to $\mathbf{r}_{\perp}$ in the plane of rotation:

$\mathbf{v} = \mathbf{n}\times\mathbf{r}_{\perp} = \mathbf{n}\times\mathbf{r}$

so

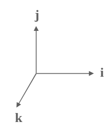
$R\mathbf{r}_{\perp} = (\cos\theta)\mathbf{r}_{\perp} + (\sin\theta)\mathbf{v}$

so

$$\begin{aligned} R\mathbf{r} &= R\mathbf{r}_{\parallel} + R\mathbf{r}_{\perp} \\ &= \mathbf{r}_{\parallel} + (\cos\theta)\mathbf{r}_{\perp} + (\sin\theta)\mathbf{v} \\ &= (\mathbf{n}\cdot\mathbf{r})\mathbf{n} + (\cos\theta)(\mathbf{r} - (\mathbf{n}\cdot\mathbf{r})\mathbf{n}) + (\sin\theta)\mathbf{n}\times\mathbf{r} \\ &= (\cos\theta)\mathbf{r} + (1-\cos\theta)(\mathbf{n}\cdot\mathbf{r})\mathbf{n} + (\sin\theta)\mathbf{n}\times\mathbf{r} \end{aligned}$$

Quaternions

- We can implement the same Euler rotation by a quaternion
- A quaternion q is a four-vector consisting of a scalar s and a three-vector $\mathbf{v}$:



$q = s + x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (s, \mathbf{v})$

Shorthand form, $\mathbf{v}=(x,y,z)$

- $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ form an orthonormal coordinate system.

Quaternions

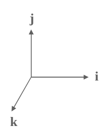
$q = (s, \mathbf{v})$

$q' = (s', \mathbf{v}')$

- Extension of complex plane:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$$

$$\left.\begin{aligned} \mathbf{ij} &= -\mathbf{ji} = \mathbf{k} \\ \mathbf{jk} &= -\mathbf{kj} = \mathbf{i} \\ \mathbf{ki} &= -\mathbf{ik} = \mathbf{j} \end{aligned}\right\} \text{cf. vector product!}$$



- Addition:

$q + q' = (s + s', \mathbf{v} + \mathbf{v}')$

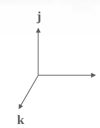
- Multiplication (quaternion product):

$qq' = (ss' - \mathbf{v}\cdot\mathbf{v}', \mathbf{v}\times\mathbf{v}' + s\mathbf{v}' + s'\mathbf{v})$

Exercise

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$$

$$\begin{aligned} \mathbf{ij} &= -\mathbf{ji} = \mathbf{k} \\ \mathbf{jk} &= -\mathbf{kj} = \mathbf{i} \\ \mathbf{ki} &= -\mathbf{ik} = \mathbf{j} \end{aligned}$$



$q = (s, \mathbf{v})$

$q' = (s', \mathbf{v}')$

$\mathbf{v} = (x, y, z)$

$\mathbf{v}' = (x', y', z')$

- Verify the first part of the multiplication rule:

$qq' = (ss' - \mathbf{v}\cdot\mathbf{v}', \mathbf{v}\times\mathbf{v}' + s\mathbf{v}' + s'\mathbf{v})$

Solution

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$$

$$\begin{aligned} \mathbf{ij} &= -\mathbf{ji} = \mathbf{k} \\ \mathbf{jk} &= -\mathbf{kj} = \mathbf{i} \\ \mathbf{ki} &= -\mathbf{ik} = \mathbf{j} \end{aligned}$$

$qq' = (s + x\mathbf{i} + y\mathbf{j} + z\mathbf{k})(s' + x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k})$

$= ss' + sx'\mathbf{i} + sy'\mathbf{j} + sz'\mathbf{k} + x\mathbf{i}s' + x\mathbf{i}x'\mathbf{i} + x\mathbf{i}y'\mathbf{j} + \ldots$

$x\mathbf{i}x'\mathbf{i} = -xx'$

Fishing out all the scalar terms gives

$ss' - xx' - yy' - zz'$

which equals

$ss' - \mathbf{v}\cdot\mathbf{v}'$

Quaternions

- The *conjugate* $\bar{q}$ of a quaternion $q = (s, \mathbf{v})$ is

$\bar{q} = (s, -\mathbf{v})$

- The *magnitude* of q is defined as

$|q| = \sqrt{s^2 + x^2 + y^2 + z^2}$

Cf. complex numbers!

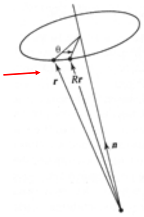
Quaternions

- *Unit* quaternions have $|q|=1$
- For a unit quaternion $q=(s,\mathbf{v})$ there exits a unit three-vector n and an angle θ such that

  $q=(\cos\theta, \sin\theta\ \mathbf{n})$

Cf. complex numbers! 19

Quaternions



- We return to our original problem
- Define quaternion $p=(0,\mathbf{r})$
- Define $R_q(p)=qp\bar{q}$
- For a unit quaternion $q=(s,\mathbf{v})$ this gives

$$R_q(p)=(0, \cos(2\theta)\mathbf{r}+(1-\cos(2\theta))(\mathbf{n}\cdot\mathbf{r})\mathbf{n}+\sin(2\theta)(\mathbf{n}\times\mathbf{r}))$$

Compare this to earlier result

$$(\cos\theta)\mathbf{r}+(1-\cos\theta)(\mathbf{n}\cdot\mathbf{r})\mathbf{n}+(\sin\theta)(\mathbf{n}\times\mathbf{r})$$

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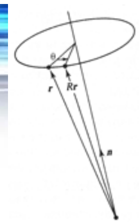
Quaternions



- Resumé:

  Rotating $\mathbf{r}$ around unit axis $\mathbf{n}$ by an angle θ can be done performing the operation $q(\cdot)\bar{q}$ on the quaternion $(0,\mathbf{r})$ where $q=(\cos(\frac{\theta}{2}), \sin(\frac{\theta}{2})\mathbf{n})$

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Quaternions

- Multiple rotations can be concatenated using quaternion multiplication:

  First 'q', then 'p' is identical to 'pq'

  $p(q(\cdot)\bar{q})\bar{p}=$

  $pq(\cdot)\bar{q}\ \bar{p}=$

  $pq(\cdot)\overline{pq}$

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Quaternions

- Property: a rotation represented by the quaternion q is also represented by $-q$

  This follows directly from the $q(.)\bar{q}$ formulation

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Exercise

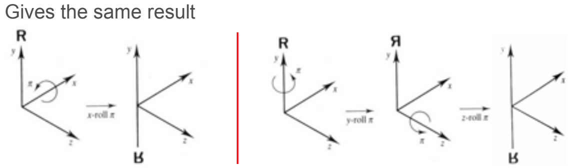
$qq'=(ss'-\mathbf{v}\cdot\mathbf{v}', \mathbf{v}\times\mathbf{v}'+s\mathbf{v}'+s'\mathbf{v})$

Verify (using quaternions) that

(1) A rotation of π around the x-axis, and

(2) A rot.of π around the y-axis and π around the z-axis

Gives the same result



Rotating $\mathbf{r}$ around unit axis $\mathbf{n}$ by an angle θ can be done performing the operation $q(\cdot)\bar{q}$ on the quaternion $(0,\mathbf{r})$ where $q=(\cos(\frac{\theta}{2}), \sin(\frac{\theta}{2})\mathbf{n})$

Multiple rotations can be concatenated using quaternion multiplication

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Solution

$qq' = (ss' - \mathbf{v}\cdot\mathbf{v}', \mathbf{v}\times\mathbf{v}' + s\mathbf{v}' + s'\mathbf{v})$

- x-roll of π corresponds to
  $(\cos(\pi/2), \sin(\pi/2)(1,0,0))$
  $= (0,(1,0,0))$
- y-roll and z-roll of π correspond to $(0,(0,1,0))$ and $(0,(0,0,1))$ respectively
- Concatenation:
  $(0,(0,0,1))(0,(0,1,0)) =$
  $(0,(0,0,1)\times(0,1,0)) =$
  $(0,(-1,0,0))$

= Identical effects, since q and $-q$ have the same effect

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Quaternions: the hard part

- We used

$$q p \bar{q} = (0, \cos(2\theta)\mathbf{r} + (1-\cos(2\theta))(\mathbf{n}\cdot\mathbf{r})\mathbf{n} + \sin(2\theta)(\mathbf{n}\times\mathbf{r}))$$

With $p = (0,\mathbf{r})$

$q = (s,\mathbf{v}) = (\cos\theta, \mathbf{n}\sin\theta)$ (unit)

To relate quaternions and rotations.

But how is this derived? We need

$$q p \bar{q} = (0, (s^2 - \mathbf{v}\cdot\mathbf{v})\mathbf{r} + 2\mathbf{v}(\mathbf{v}\cdot\mathbf{r}) + 2s(\mathbf{v}\times\mathbf{r}))$$

Because by substituting q this leads to

$$q p \bar{q} = (0, (\cos^2\theta - \sin^2\theta)\mathbf{r} + 2\sin^2\theta\,\mathbf{n}(\mathbf{n}\cdot\mathbf{r}) + 2\cos\theta\sin\theta(\mathbf{n}\times\mathbf{r}))$$

$$= (0, \mathbf{r}\cos 2\theta + (1-2\cos 2\theta)\mathbf{n}(\mathbf{n}\cdot\mathbf{r}) + \sin 2\theta(\mathbf{n}\times\mathbf{r}))$$

=

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Quaternions: the hard part

- We need:

$$q p \bar{q} = (0, (s^2 - \mathbf{v}\cdot\mathbf{v})\mathbf{r} + 2\mathbf{v}(\mathbf{v}\cdot\mathbf{r}) + 2s(\mathbf{v}\times\mathbf{r}))$$

With $p = (0,\mathbf{r})$

$q = (s,\mathbf{v})$ (unit)

- [Watt (p.381) says this is "easily shown"]
- Exercise: show it!

$qq' = (ss' - \mathbf{v}\cdot\mathbf{v}', \mathbf{v}\times\mathbf{v}' + s\mathbf{v}' + s'\mathbf{v})$

$\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$
$\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b} + \mathbf{a}\times\mathbf{c}$
$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$
$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b})$
$[(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = -\mathbf{c}\times(\mathbf{a}\times\mathbf{b})]$

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Solution

$qq' = (ss' - \mathbf{v}\cdot\mathbf{v}', \mathbf{v}\times\mathbf{v}' + s\mathbf{v}' + s'\mathbf{v})$

$p = (0,\mathbf{r})$
$q = (s,\mathbf{v})$ (unit)

$\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$
$\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b} + \mathbf{a}\times\mathbf{c}$
$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$
$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b})$
$[(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = -\mathbf{c}\times(\mathbf{a}\times\mathbf{b})]$

$$q p \bar{q} = (-\mathbf{v}\cdot\mathbf{r}, \mathbf{v}\times\mathbf{r} + s\mathbf{r})\bar{q}$$
$$= (-\mathbf{v}\cdot\mathbf{r}, \mathbf{v}\times\mathbf{r} + s\mathbf{r})(s, -\mathbf{v})$$
$$= ((-\mathbf{v}\cdot\mathbf{r})s - (\mathbf{v}\times\mathbf{r} + s\mathbf{r})\cdot(-\mathbf{v}), (\mathbf{v}\times\mathbf{r} + s\mathbf{r})\times(-\mathbf{v}) + (-\mathbf{v}\cdot\mathbf{r})(-\mathbf{v}) + s(\mathbf{v}\times\mathbf{r} + s\mathbf{r}))$$

$$= -s\mathbf{v}\cdot\mathbf{r} + (\mathbf{v}\times\mathbf{r})\cdot\mathbf{v} + s\mathbf{v}\cdot\mathbf{r} = 0$$

$= 0$ because $(\mathbf{v}\times\mathbf{r}) \perp \mathbf{v}$

$$= \mathbf{v}\times(\mathbf{v}\times\mathbf{r} + s\mathbf{r}) + (\mathbf{v}\cdot\mathbf{r})\mathbf{v} + s(\mathbf{v}\times\mathbf{r}) + s^2\mathbf{r}$$
$$= \mathbf{v}\times(\mathbf{v}\times\mathbf{r}) + \mathbf{v}\times(s\mathbf{r}) + (\mathbf{v}\cdot\mathbf{r})\mathbf{v} + s(\mathbf{v}\times\mathbf{r}) + s^2\mathbf{r}$$
$$= \mathbf{v}(\mathbf{v}\cdot\mathbf{r}) - \mathbf{r}(\mathbf{v}\cdot\mathbf{v}) + s(\mathbf{v}\times\mathbf{r}) + (\mathbf{v}\cdot\mathbf{r})\mathbf{v} + s(\mathbf{v}\times\mathbf{r}) + s^2\mathbf{r}$$
$$= (s^2 - \mathbf{v}\cdot\mathbf{v})\mathbf{r} + 2\mathbf{v}(\mathbf{v}\cdot\mathbf{r}) + 2s(\mathbf{v}\times\mathbf{r})$$

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Interpolating quaternions

- Keyframe interpolation of object orientation:
  1. Animator specifies orientation using (eventually) a rotation matrix at keyframes
  2. Matrices are converted to quaternions
  3. Quaternion sequence is interpolated for each frame
  4. Each frame quaternion is converted to a rotation matrix

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Rotation matrix/quaternion conversion

$q = (\cos(\theta/2), \sin(\theta/2)\mathbf{n}) = (s,(x,y,z))$

$$\mathbf{M} = \begin{pmatrix} 1-2(y^2+z^2) & 2xy-2sz & 2sy+2xz & 0 \\ 2xy+2sz & 1-2(x^2+z^2) & -2sx+2yz & 0 \\ -2sy+2xz & 2sx+2yz & 1-2(x^2+y^2) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{M} = \begin{pmatrix} M_{00} & M_{01} & M_{02} & 0 \\ M_{10} & M_{11} & M_{12} & 0 \\ M_{20} & M_{21} & M_{22} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{cases} q = (s,(x,y,z)) \\ s = \frac{1}{2}\sqrt{M_{00} + M_{11} + M_{22} + 1} \\ x = \frac{1}{4s}(M_{21} - M_{12}) \\ y = \frac{1}{4s}(M_{02} - M_{20}) \\ z = \frac{1}{4s}(M_{10} - M_{01}) \end{cases}$$

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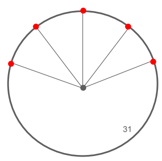
Quaternion interpolation

- Rotations correspond to unit quaternions
- Quaternions are four-vectors

→ A rotation is a point on a 4D unit hypersphere

- Interpolation of quaternions means interpolating a path *on* this hypersphere
- *not* through it, which happens with linear interpolation and gives unwanted speed changes

2D analogue



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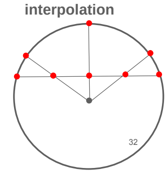
Quaternion interpolation

- Rotations correspond to unit quaternions
- Quaternions are four-vectors

→ A rotation is a point on a 4D unit hypersphere

- Interpolation of quaternions means interpolating a path *on* this hypersphere
- *not* through it, which happens with linear interpolation and gives unwanted speed changes

2D analogue: wrong interpolation



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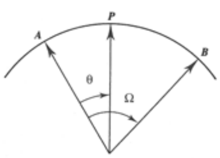
Quaternion interpolation

- The path *on* the hypersphere is found using *spherical linear interpolation* (slerp)



$\mathbf{P} = \alpha\mathbf{A} + \beta\mathbf{B}$

α and β solved from

$$\left.\begin{array}{l} |\mathbf{P}| = \mathbf{1} \\ \mathbf{A}\cdot\mathbf{B} = \cos\Omega \\ \mathbf{A}\cdot\mathbf{P} = \cos\theta \end{array}\right\} \mathbf{P} = \mathbf{A}\frac{\sin(\Omega-\theta)}{\sin\Omega} + \mathbf{B}\frac{\sin\theta}{\sin\Omega}$$

with variable $u \in [0,1]$ instead of θ: $\mathbf{P} = \mathbf{A}\frac{\sin((1-u)\Omega)}{\sin\Omega} + \mathbf{B}\frac{\sin(u\Omega)}{\sin\Omega}$

generalises to quaternions:

$$\text{slerp}(q_1, q_2, u) = q_1\frac{\sin((1-u)\Omega)}{\sin\Omega} + q_2\frac{\sin(u\Omega)}{\sin\Omega}$$

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Quaternion interpolation

- Between any two unit quaternions p and q there exist *two* spherical paths
- Problem: *slerp* may not chose the shortest path
- Solution (see Watt 13.6.3 for details):

  if $(p-q)\cdot(p-q) < (p+q)\cdot(p+q)$ use

  $\text{slerp}(p,q,u)$

  else use $\text{slerp}(p,-q,u)$

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