Curves and Surfaces

Bézier curves and B-splines

Overview

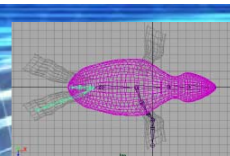
- Curves: analytic, Bézier, B-spline, rational, NURBS
- Surfaces
- Modelling with patches
  - Sweep surfaces
  - Patch editing
  - Surface fitting
- Objects

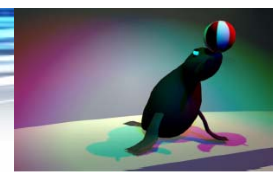
Curves and surfaces in modelling

Bi-cubic (and higher order) patches are an alternative to polygon meshes

- Inherently smooth
- Natural for hand-editing
- Shape change animation potential
- Compact representation
- Potentially less CPU-GPU traffic
- LOD easier
- Exact (CAD!)

- Less popular → less dedicated hardware
- Complex (maths)
- In general slower rendering



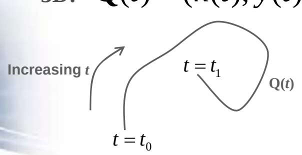


Cubic (and higher order) patches

- State of the art in CAD, not in games / mainstream graphics
- Exact in mathematical terms, but shape of patches is still limited. (e.g. Bézier form cannot represent circle) → approximation

Analytical curves

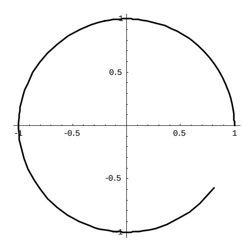
- Curve **Q**, with one parameter t
  - 2D: $\mathbf{Q}(t) = (x(t), y(t))$
  - 3D: $\mathbf{Q}(t) = (x(t), y(t), z(t))$



Example

$\mathbf{Q}(t) = (1, t)$

$t \in [0,10]$



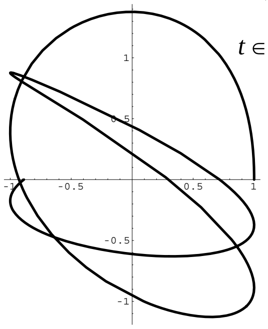
$\mathbf{Q}(t) = (\cos(t), \sin(t))$

$t \in [0, 1.8\pi]$

Example

$$\mathbf{Q}(t) = \left(\cos(t^{1.1}), \sin(t)\left(\tfrac{3}{2} - \frac{t}{4\pi}\right)\right)$$

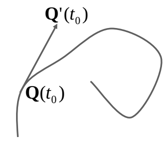
$t \in [0, 4\pi]$



Tangent vector

- The tangent vector at any point t_0 :

$$\mathbf{Q}'(t_0) = (x'(t_0), y'(t_0), z'(t_0))$$

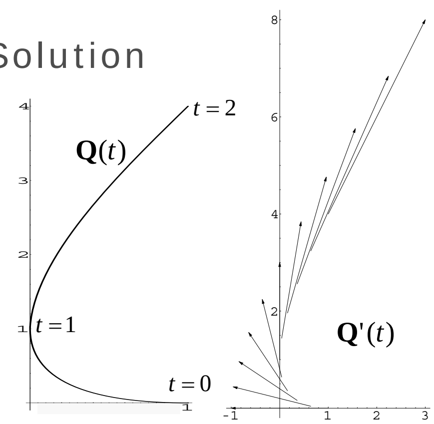


- Tangent line equation: $\mathbf{Q}(t_0) + r\mathbf{Q}'(t_0)$

Exercise

1. Draw the curve, $\mathbf{Q}(t) = \left((t-1)^2, t^2\right)$, with $t \in [0,2]$
2. Compute and draw some of the tangent vectors

Solution



Reparametrisation to arc length

- It may be that small variations in t gives large variations in $\mathbf{Q}$, and vice versa, even in one curve
- Solution: reparametrise $\mathbf{Q}(t)$ to $\mathbf{Q}(s)$, where s is arc length:

$$s = s(t) = \int_{t_0}^{t} \sqrt{x'^2 + y'^2}\, d\tau$$

Example

$\mathbf{Q}(t) = (t, t), \qquad t \in [0, \infty\rangle$

$$s(t) = \int_0^t \sqrt{1+1}\, d\tau = \sqrt{2}t$$

Then

Travels $\sqrt{2}$

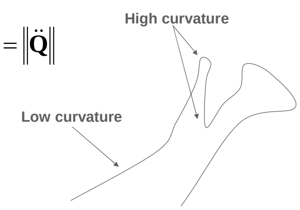
Travels 1

$\mathbf{Q}(0) = (0,0)$	$t = 0$	$s = 0$
$\mathbf{Q}(1) = (1,1)$	$t = 1$	$s = \sqrt{2}$
$\mathbf{Q}(\frac{1}{\sqrt{2}}) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$	$t = \frac{1}{\sqrt{2}}$	$s = 1$

Curvature

- The curvature of a (2D or 3D) curve equals

$$\kappa = \left\| \frac{d^2\mathbf{Q}}{ds^2} \right\| = \|\ddot{\mathbf{Q}}\|$$



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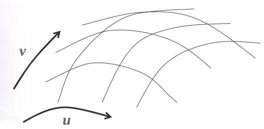
Analytic surface

- Surface $\mathbf{Q}$ is defined by two parameters u and v:

$$\mathbf{Q} = \mathbf{Q}(u,v) = (x(u,v), y(u,v), z(u,v))$$



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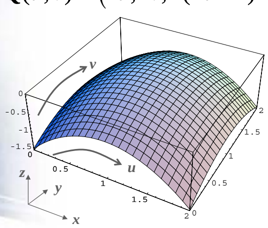
Example

$$\mathbf{Q}(u,v) = \left(2u, 2v, -(2u-1)^2 - \tfrac{1}{2}(2v-1)^2\right)$$

$$u, v \in [0,1]$$



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Tangent vectors

- The tangent vectors in the u and v directions can be computed by the partial derivatives $\mathbf{Q}_u$ and $\mathbf{Q}_v$:

$$\mathbf{Q}_u = \frac{\partial \mathbf{Q}}{\partial u} \qquad \mathbf{Q}_v = \frac{\partial \mathbf{Q}}{\partial v}$$

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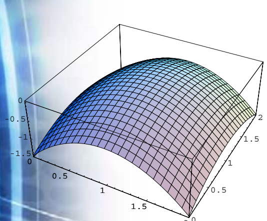
Exercise

$$\mathbf{Q}(u,v) = \left(2u, 2v, -(2u-1)^2 - \tfrac{1}{2}(2v-1)^2\right)$$

$$u, v \in [0,1]$$



Compute the partial derivatives $\mathbf{Q}_u$ and $\mathbf{Q}_v$ and verify them for the (u, v) pairs

$(\tfrac{1}{2}, 0)$

$(0, \tfrac{1}{2})$

$(1, \tfrac{1}{2})$

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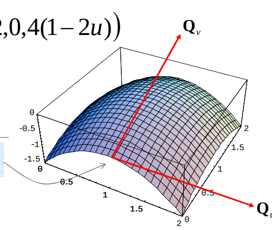
Solution

$$\mathbf{Q}_u = \left(2, 0, -2(2u-1)\cdot 2\right) = \left(2, 0, 4(1-2u)\right)$$

$$\mathbf{Q}_v = \left(0, 2, 2(1-2v)\right)$$

(u,v)	$\mathbf{Q}_u$	$\mathbf{Q}_v$
$(\tfrac{1}{2}, 0)$	$(2,0,0)$	$(0,2,2)$
$(0, \tfrac{1}{2})$	$(2,0,4)$	$(0,2,0)$
$(1, \tfrac{1}{2})$	$(2,0,-4)$	$(0,2,0)$



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Normal vector

- The normal vector $\mathbf{N}$ of a surface $\mathbf{Q}$ equals

$$\mathbf{N} = \mathbf{Q}_u \times \mathbf{Q}_v$$