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Geometric constructions for generalized barycentric coordinates

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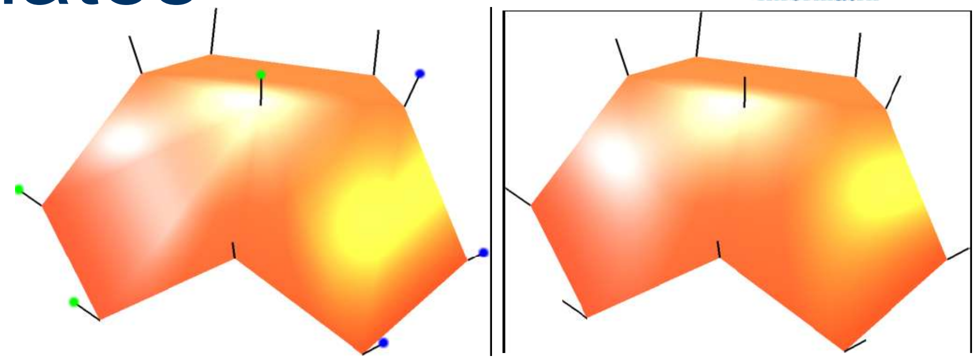
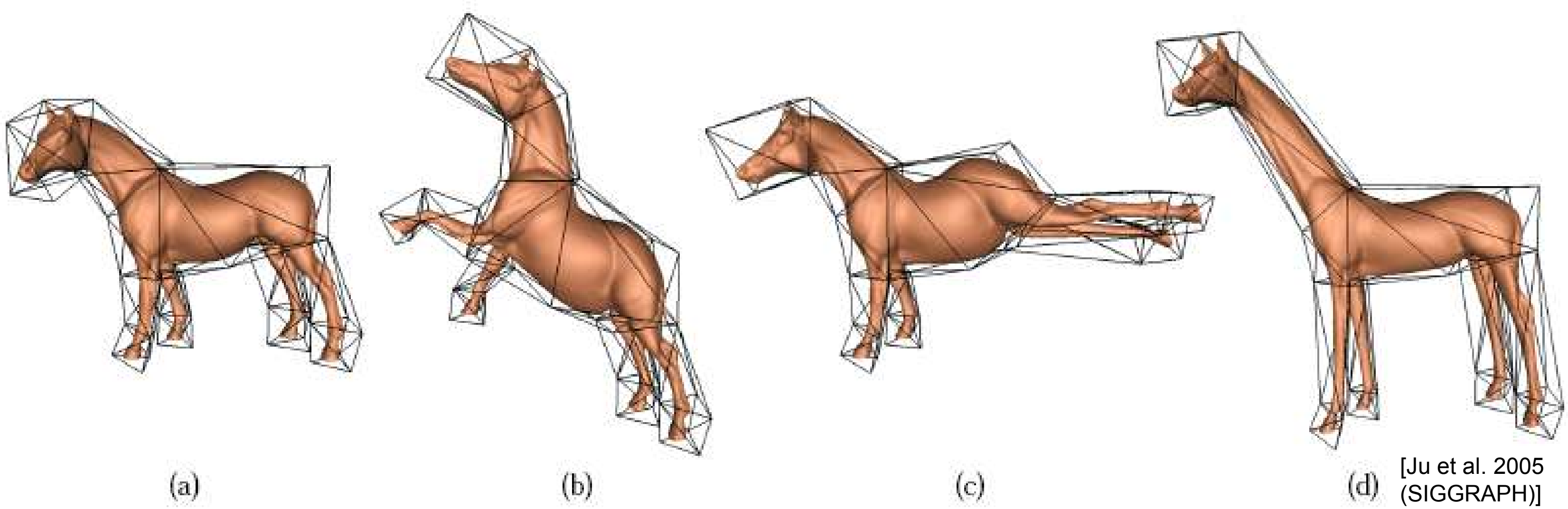
Barycentric Coordinates

Applications:

- Interpolation/Shading
- Bézier surfaces
- Free-form deformations
- Finite elements
- Parameterization



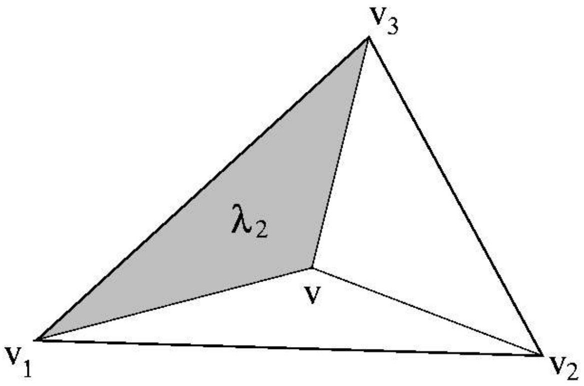
[Hormann 2005]

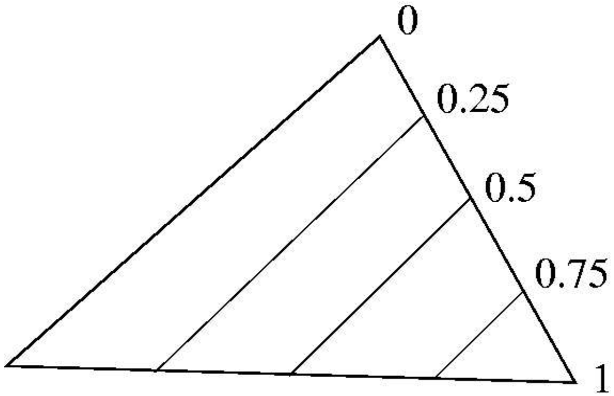


[Ju et al. 2005 (SIGGRAPH)]

Barycentric Coordinates

Properties:





$$\forall i \; \lambda_i(v) > 0 \qquad \text{positivity,}$$

$$\sum_i \lambda_i(v) = 1 \qquad \text{partition of unity,}$$

$$\sum_i \lambda_i(v) v_i = v \qquad \text{linear precision.}$$

Overview

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- Prior work
- Spherical barycentric coordinates
- 3D barycentric coordinates
- Results and conclusions

Overview

- **Prior work**
- Spherical barycentric coordinates
- 3D barycentric coordinates
- Results and conclusions

Prior work

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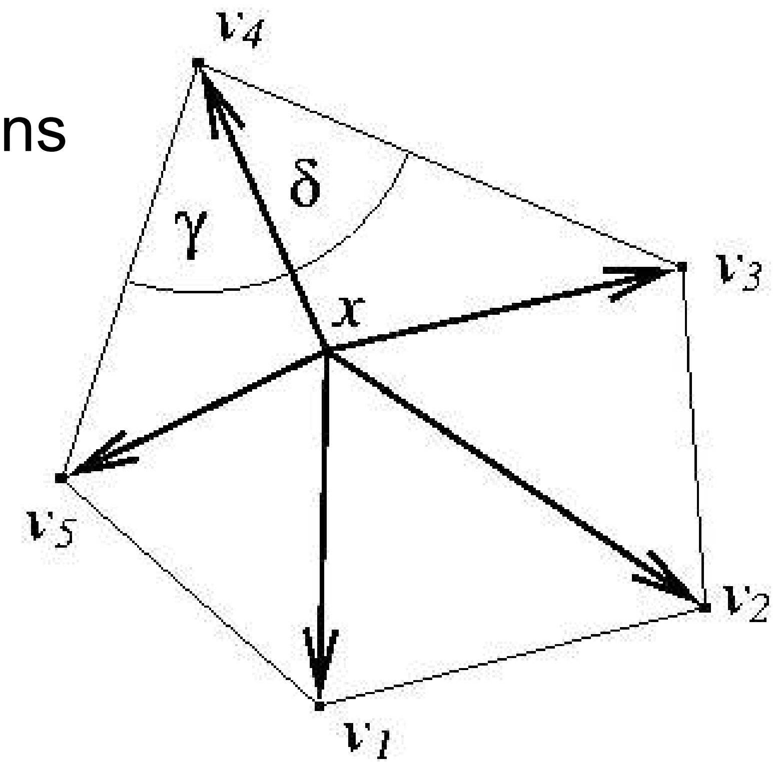
- Wachspress coordinates [Wachspress 1975, Meyer et al. 2002]
- Properties:
  - positive for convex polygons
  - only defined inside of convex polygons

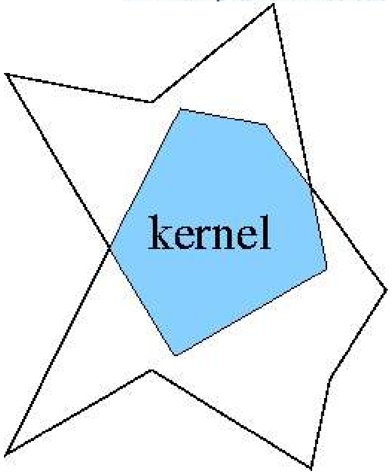
$$\lambda_i(x) = \frac{w_i(x)}{\sum_j w_j(x)}$$

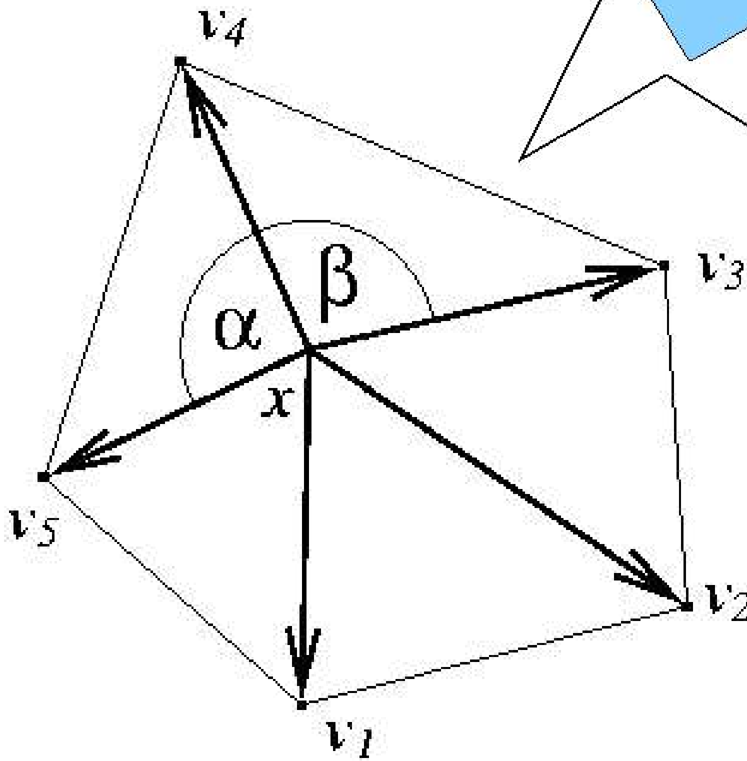
$$w_4(x) = \frac{\cot\gamma + \cot\delta}{\|v_4 - x\|^2}$$



Prior work

- Mean value coordinates [Floater 2003]
- Properties:
  - defined for arbitrary polygons
  - positive in the kernel of arbitrary polygons

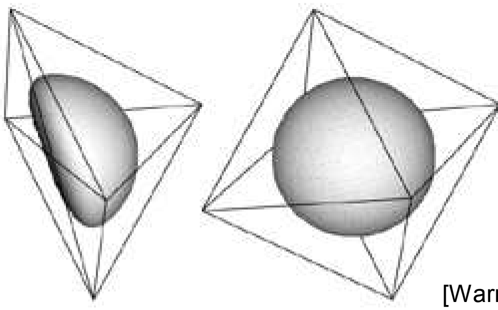




$$w_4(x) = \frac{\tan\frac{\alpha}{2} + \tan\frac{\beta}{2}}{\|v_4 - x\|}$$

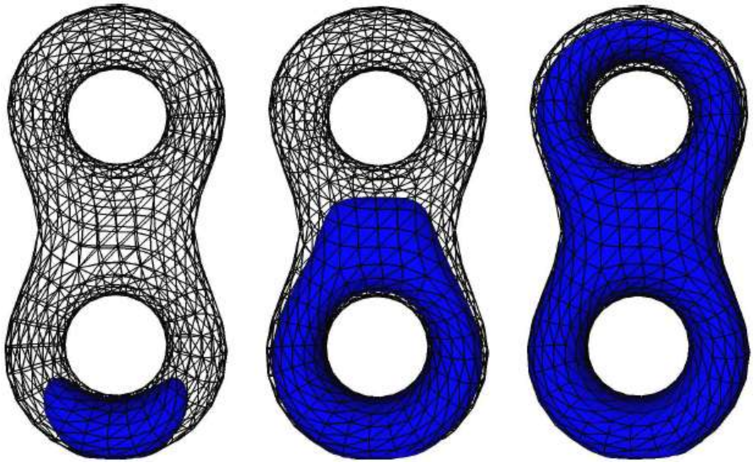
Prior work

- 3D Wachspress coordinates [Warren 1996]



[Warren 1996]

- 3D mean value coordinates [Floater et al. 2005,
Ju et al. 2005 (SIGGRAPH)]



[Floater et al. 2005]

- 3D barycentric coordinates [Ju & Warren 2006]

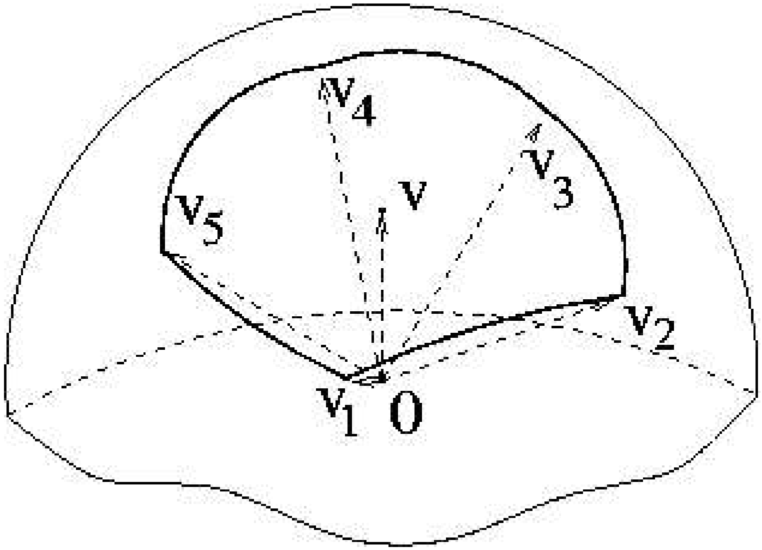
Overview

- Prior work
- **Spherical barycentric coordinates**
- 3D barycentric coordinates
- Results and conclusions

Spherical coordinates

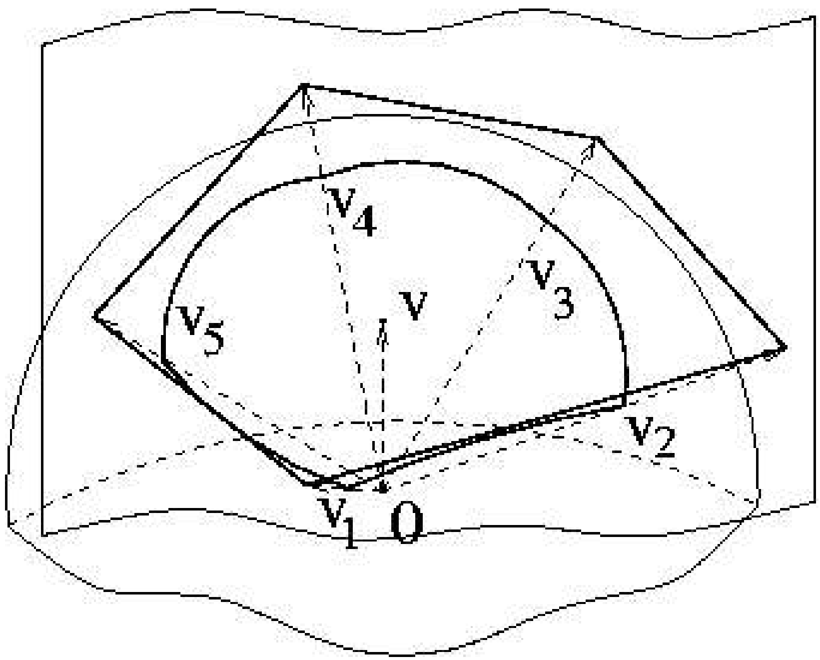
- Given a polygon on the unit sphere at the origin.
- Can we find coordinates such that

$$\sum_i \lambda_i(v) v_i = v \text{ ?}$$



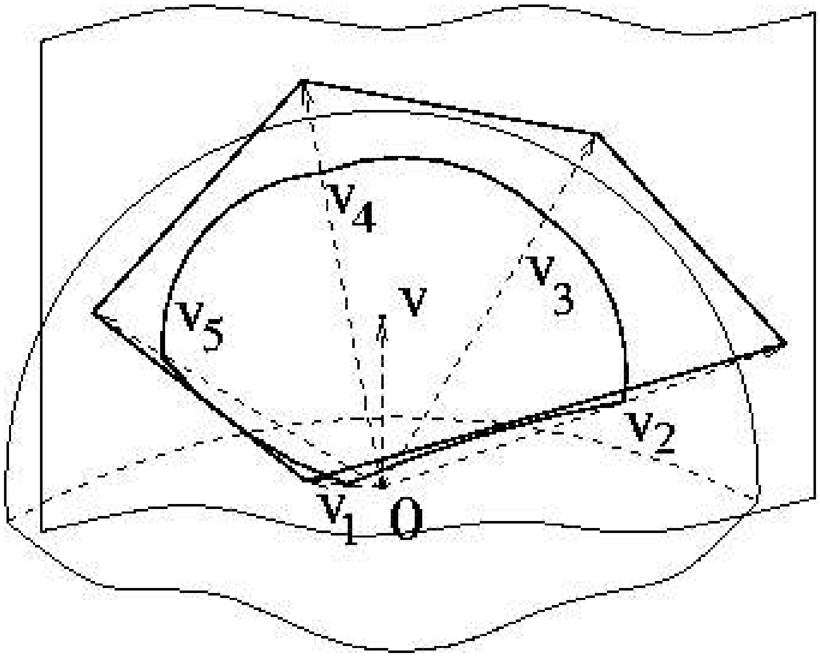
Spherical coordinates

- Idea: Project the vertices to the tangent plane at *v*.



Spherical coordinates

- Idea: Project the vertices to the tangent plane at *v*.
- This yields a planar polygon, and planar coordinates can be used.

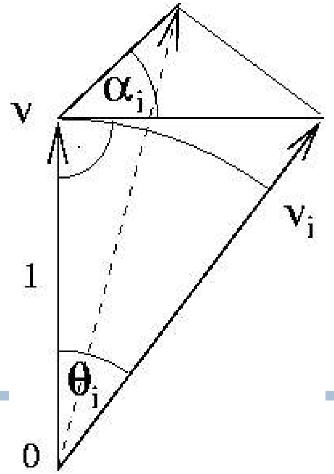


Spherical coordinates

All planar barycentric coordinates have analogues on the sphere. Interesting special cases:

- Spherical Wachspress coordinates [Ju et al. 2005 (SGP)]
- Spherical mean value coordinates

$$\lambda_i(v) = \frac{\tan\frac{\alpha_{i-1}}{2} + \tan\frac{\alpha_i}{2}}{\sin\theta_i} \Big/ \sum_j \cot\theta_j \left(\tan\frac{\alpha_{j-1}}{2} + \tan\frac{\alpha_j}{2}\right),$$



Properties

Spherical mean value coordinates are

- defined for arbitrary spherical polygons
- positive in the kernel of all spherical polygons
- $\sum_i \lambda_i(v) \geq 1$ in the kernel of all spherical polygons
- invariant under rotations

Overview

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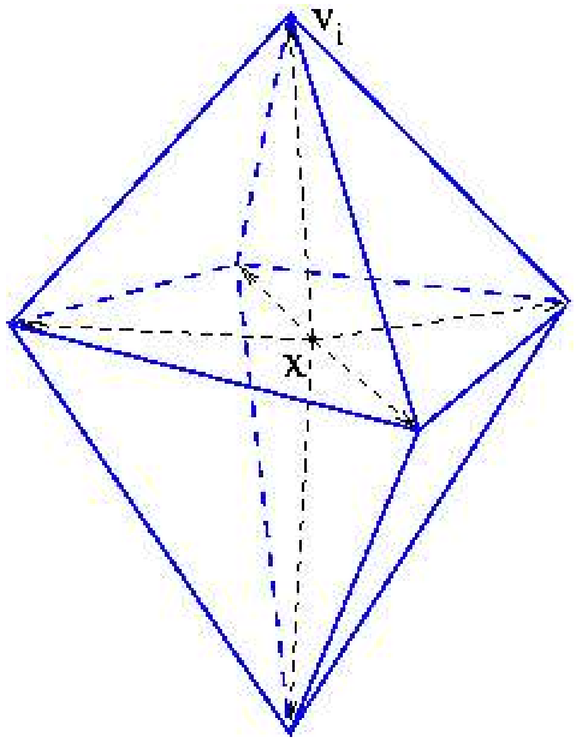
3D mean value coordinates

- General idea: Find coordinates that satisfy

$$\sum_i w_i(x) v_i = x$$

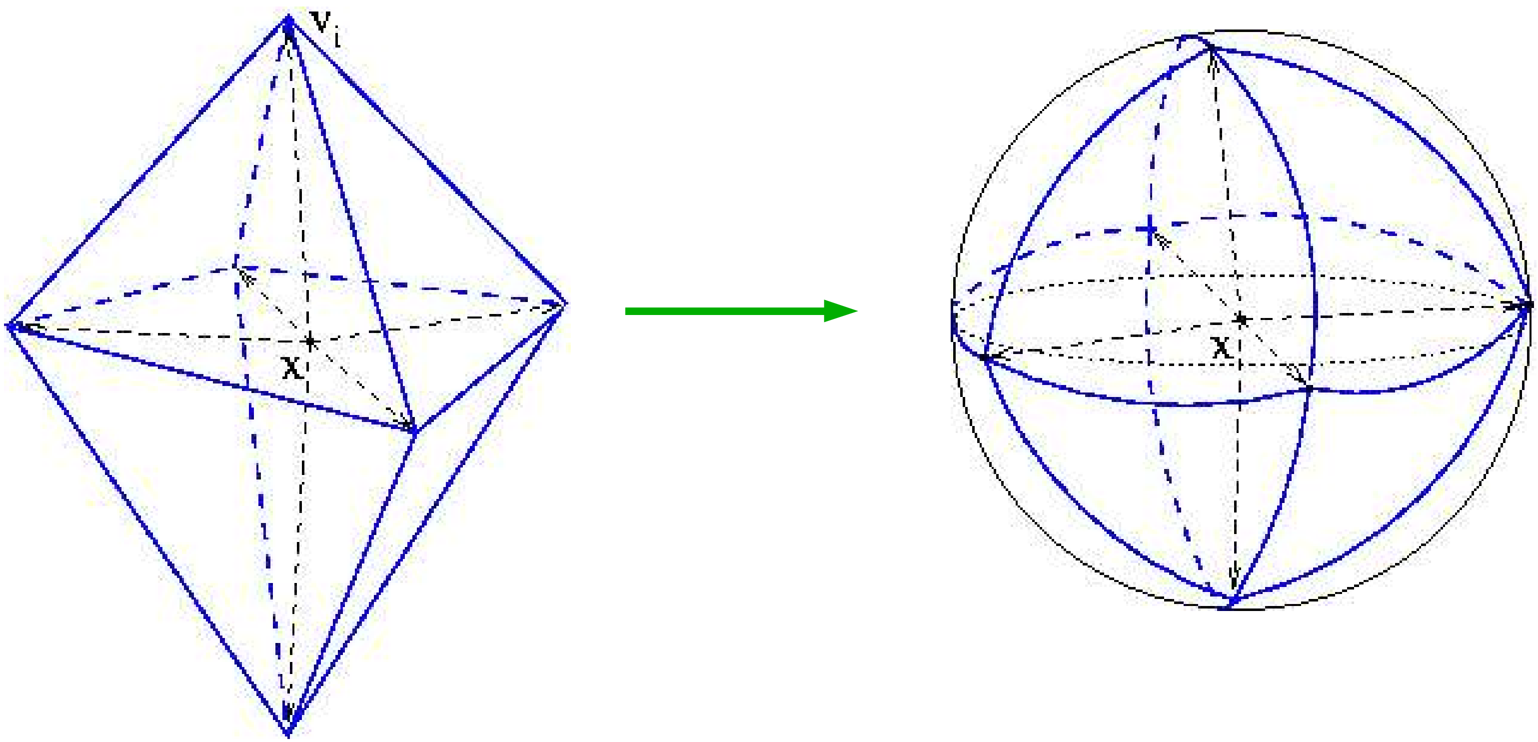
$$\Leftrightarrow \sum_i w_i(x)(v_i - x) = 0$$



3D mean value coordinates

Approach:

- Project the mesh to the unit sphere centered at ***x***.



3D mean value coordinates

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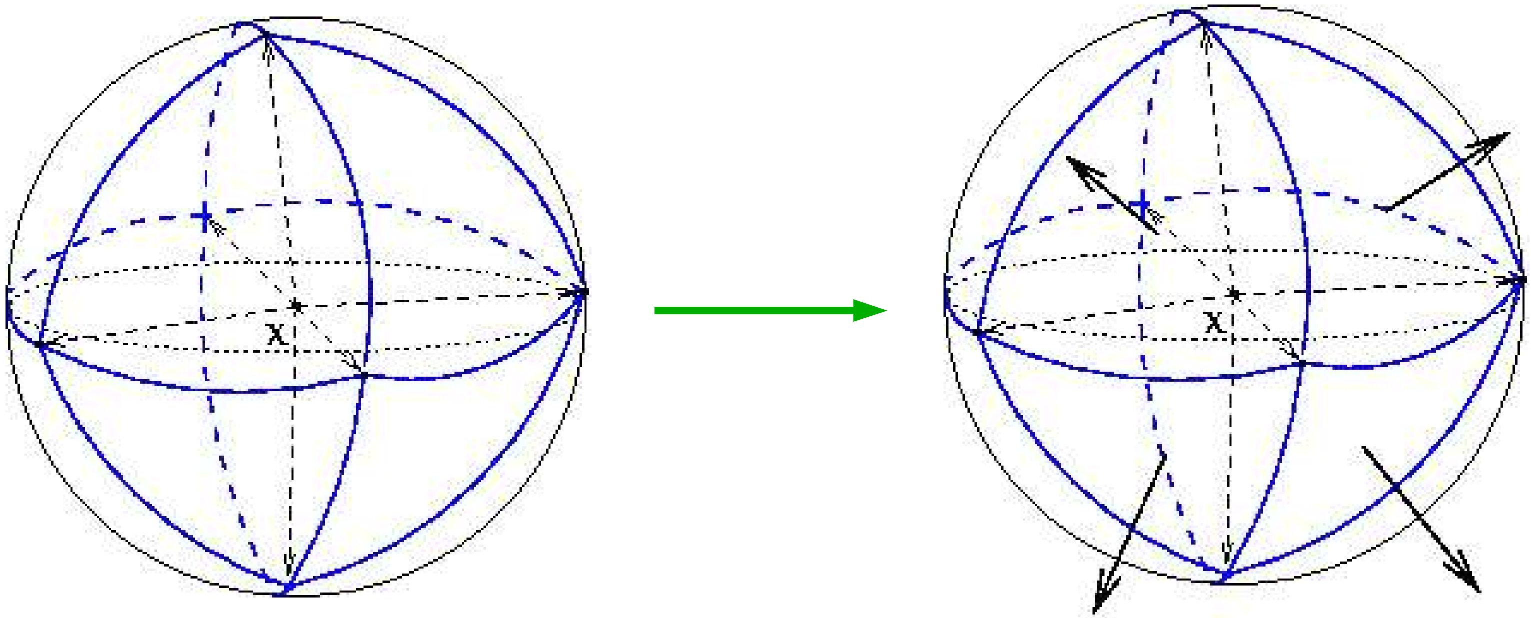
Approach:

- Project vertices to unit sphere at ***x***.
- Assign a face vector $\mathbf{v}_F$ to each face *F*.



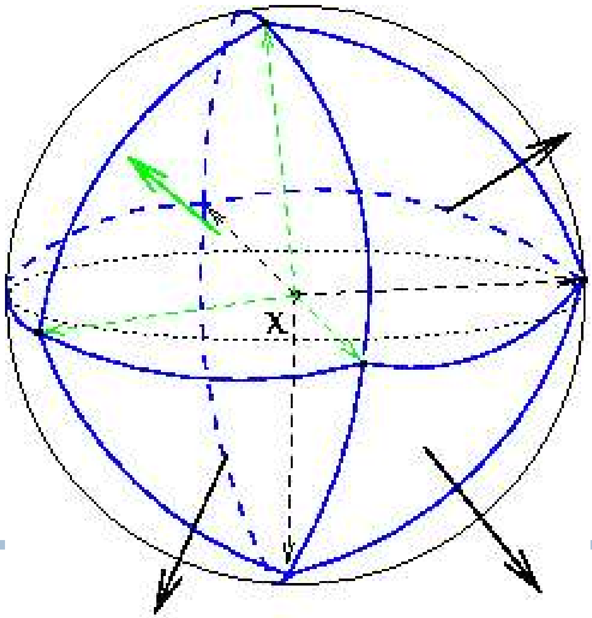
3D mean value coordinates

Approach:

- Project vertices to unit sphere at ***x***.
- Assign a face vector $\mathbf{v}_F$ to each face *F*.
- Distribute the face vectors to the vertex vectors using spherical mean value coordinates.

$$\sum_F \mathbf{v}_F = \mathbf{0}$$

$$\sum_i w_i(\mathbf{v}_i - \mathbf{x}) \stackrel{!}{=} \mathbf{0}$$

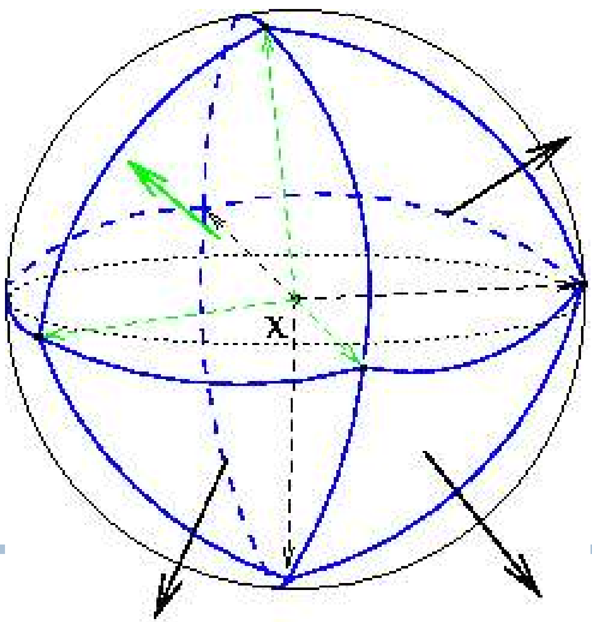


3D mean value coordinates

Approach:

- Project vertices to unit sphere at ***x***.
- Assign a face vector $\mathbf{v}_F$ to each face *F*.
- Distribute the face vectors to the vertex vectors using spherical mean value coordinates.

➔ 3D mean value coordinates are obtained.



Properties

- 3D mean value coordinates are
  - defined for arbitrary polyhedra in $\mathbb{R}^3$
  - positive inside the kernel
  - invariant under similarity transformations
  - reduce to planar mean value coordinates on the faces
- A similar construction can be used to obtain 3D Wachspress coordinates, 3D discrete harmonic coordinates, ...

Overview

- Prior work
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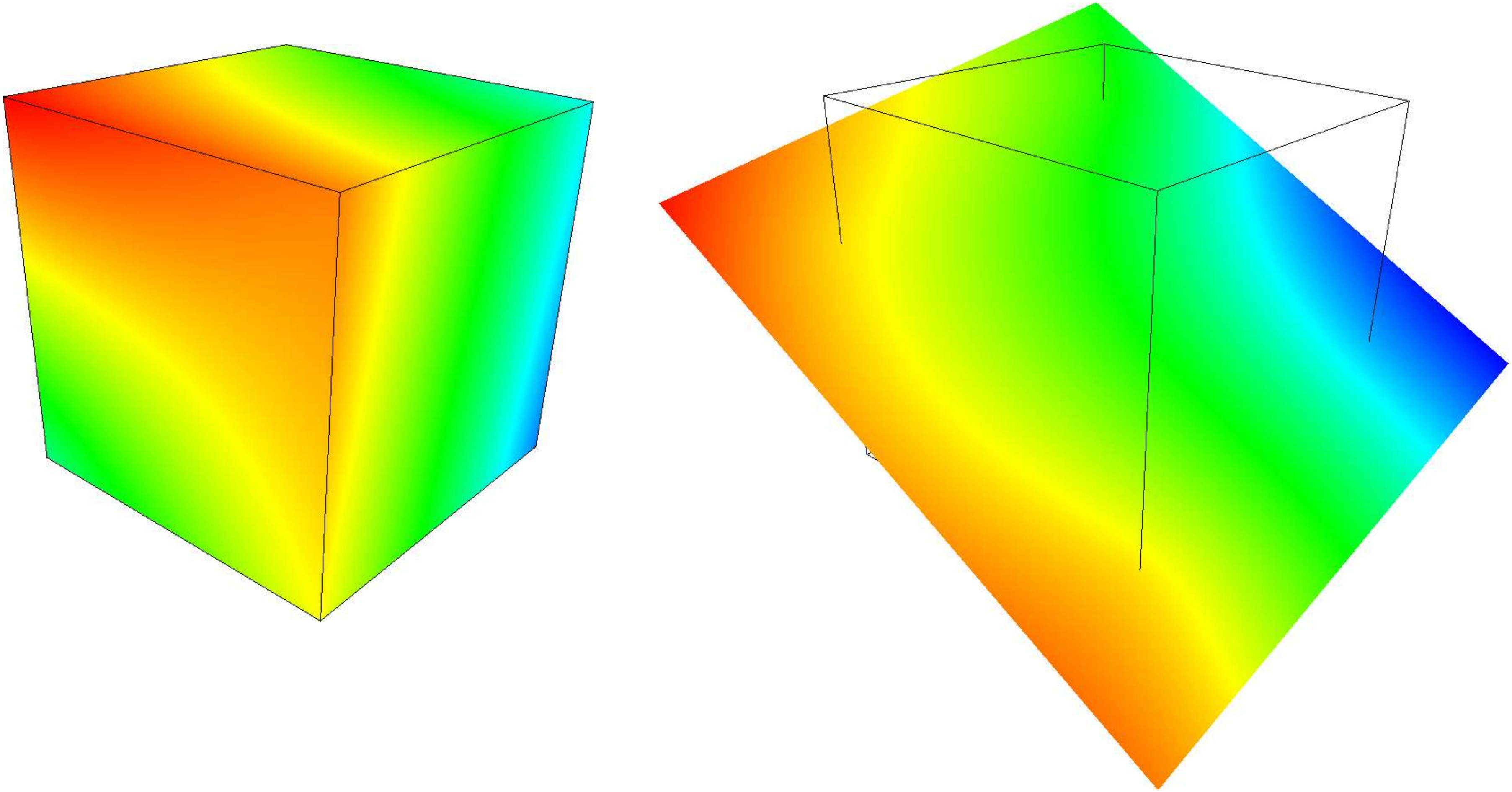
- 3D barycentric coordinates
- **Results and conclusions**

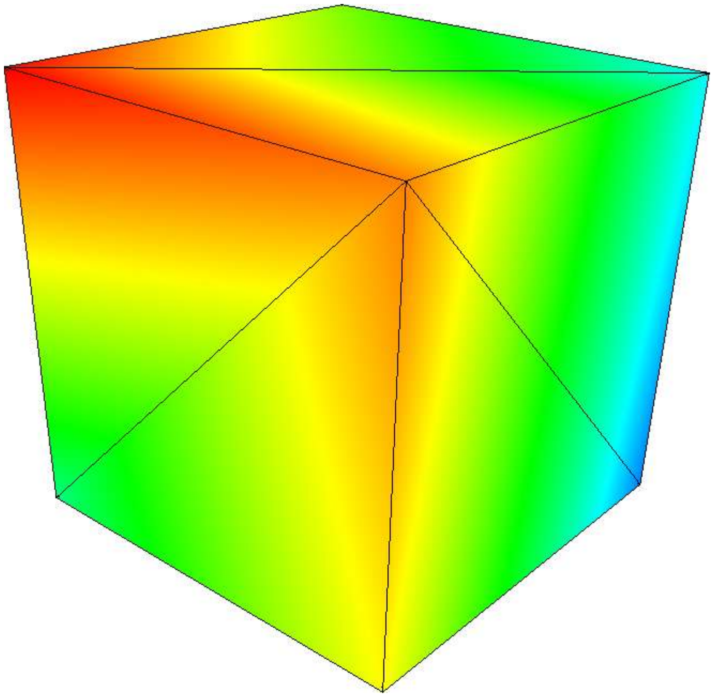
Results

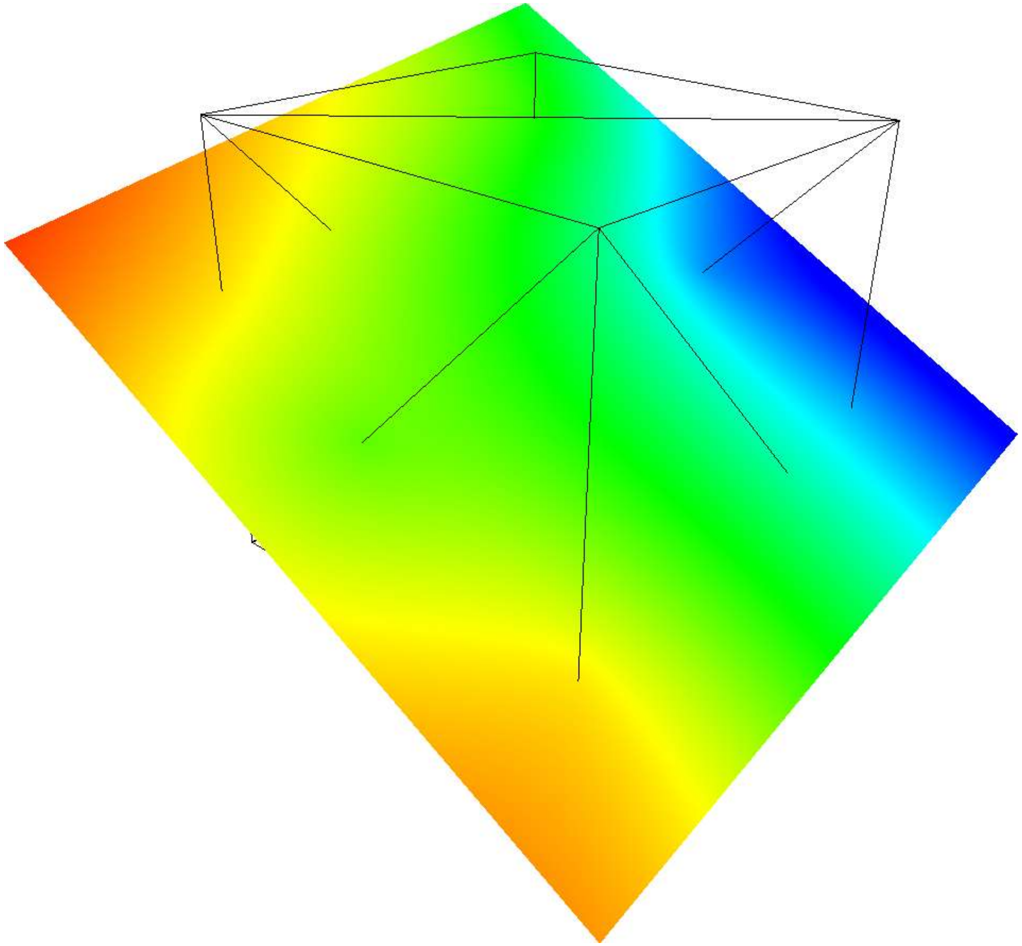
- Interpolation and extrapolation



Results

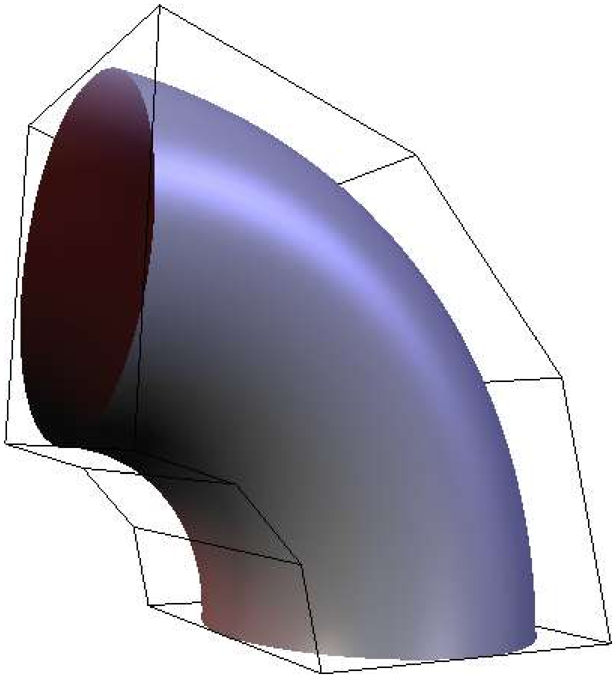
- Interpolation and extrapolation

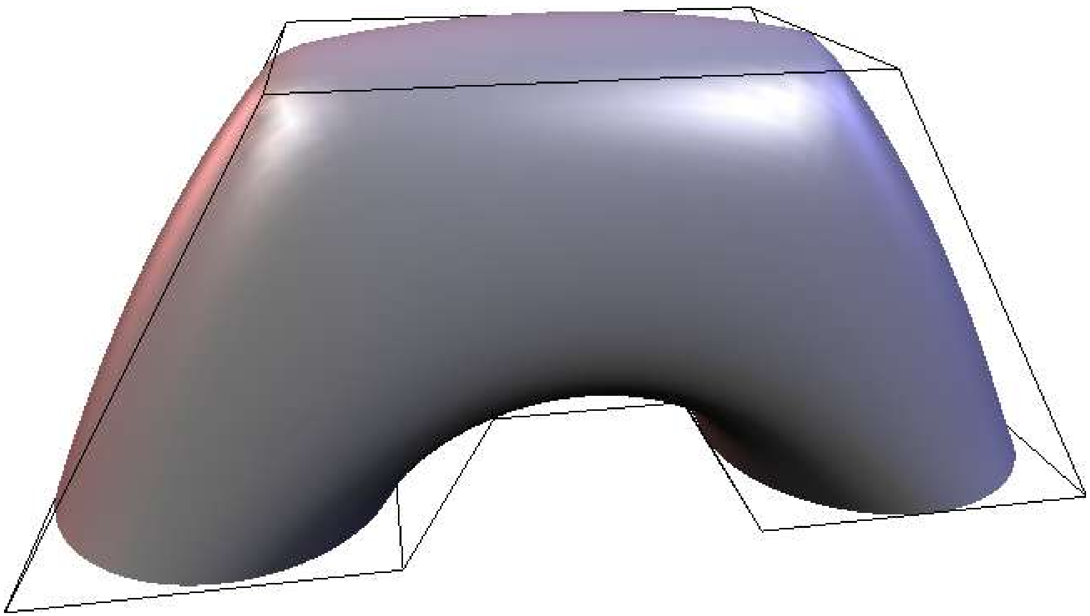




Results

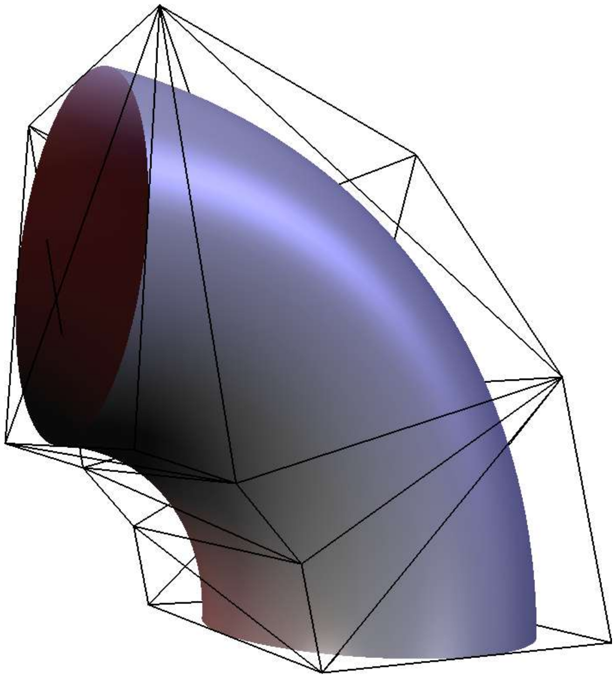
- Space deformation

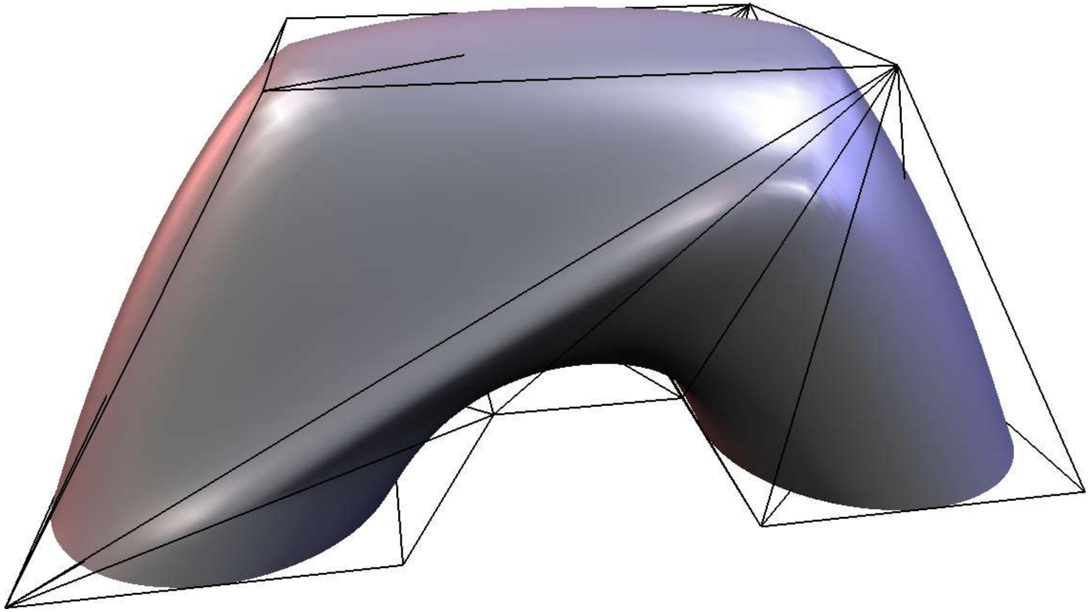




Results

- Space deformation





Conclusions

We introduced

- Spherical barycentric coordinates
  - a generalization of the vector coordinates in [Ju et al. 2005 (SGP)]
- 3D barycentric coordinates for arbitrary meshes
  - a generalization of [Floater et al. 2005, Ju et al. 2005 (SIGGRAPH), Ju & Warren 2006]
  - in particular, 3D mean value coordinates
  - this concludes the generalization of mean value coordinates from 2D to 3D

Future work

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- Relation to 3D Wachspress coordinates [Warren 1996]
- Extension to other surfaces
- Extension to sets with smooth boundary

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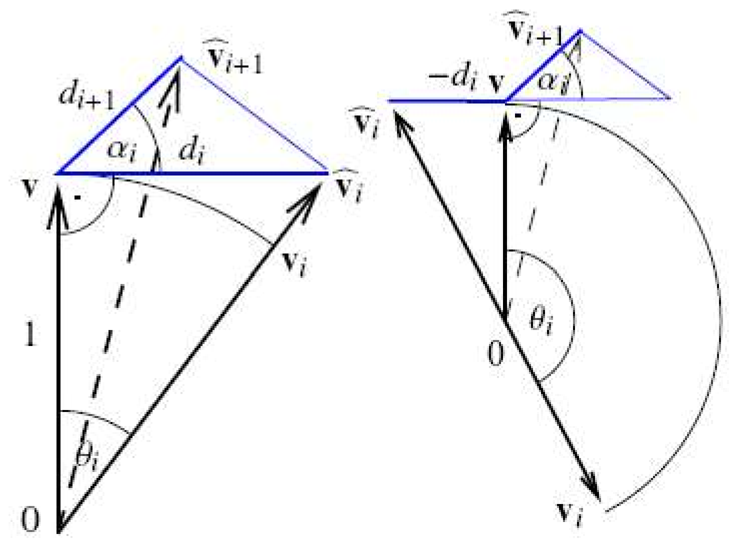
Thank you!

Spherical mean value coord.

- θ<90˚: “forward projection”
- θ>90˚: “backward projection”
  - Scaling factor tends to infinity
  - Weight tends to zero

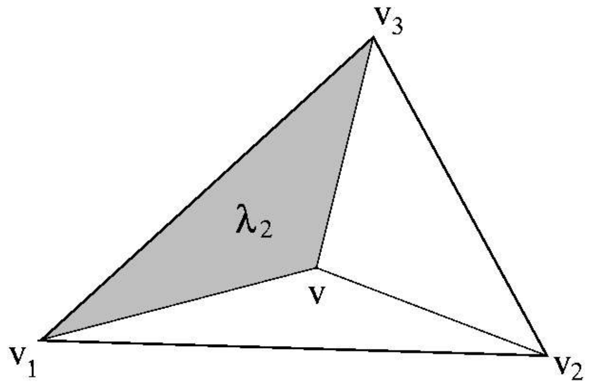
$$w(x) = \frac{\tan\frac{\alpha}{2} + \tan\frac{\beta}{2}}{\|v - x\|}$$

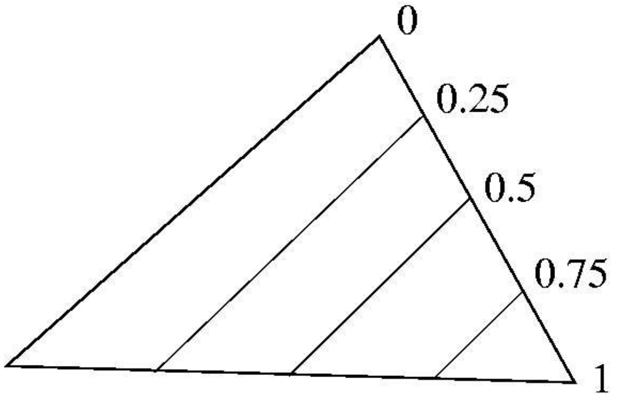


- ➔ Continous coordinates

$$\lambda_i(v) = \frac{\tan\frac{\alpha_{i-1}}{2} + \tan\frac{\alpha_i}{2}}{\sin\theta_i} \Big/ \sum_j \cot\theta_j \left(\tan\frac{\alpha_{j-1}}{2} + \tan\frac{\alpha_j}{2}\right)$$

Barycentric Coordinates





Properties:

$$\forall i \ \lambda_i(v) > 0$$ positivity,

$$\sum_i \lambda_i(v) = 1$$ partition of unity,

$$\sum_i \lambda_i(v) v_i = v$$ linear precision,

$$\Rightarrow \sum_i \lambda_i(v) f(v_i) = f(v)$$ affine interpolation.

Prior work

- Discrete harmonic coordinates [Pinkall & Polthier 1993]
- Properties:
  - yields a discrete version of a harmonic function
  - in general not positive
  - only defined inside of convex polygons

$$w_4(x) = \cot\gamma + \cot\delta$$